k-Du Bois and k-rational singularities
(joint work w/ Sridhar Venkatesh and Anh Duc Vo)

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Outline

1. Rational and Du Bois singularities

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1. Rational and Du Bois singularities
2. k-rational and k-Du Bois singularities in the local complete intersection (lci) case

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1. Rational and Du Bois singularities
2. k-rational and k-Du Bois singularities in the local complete intersection (lci) case
3. Problem outside the lci case

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Throughout, we work over the complex numbers $\mathbb{C}$.

Rational singularities

Definition (Artin 1966, Kempf 1973)

A variety X is said to have *rational singularities* if for some (any) proper birrational map $f : \tilde{X} \to X$ such that $\tilde{X}$ is smooth, the natural map

$$\mathcal{O}_X \to \mathbf{R}f_*\mathcal{O}_{\tilde{X}}$$

is a quasi-isomorphism.

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Example

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3. Singular surfaces in $\mathbb{P}^3$ with rational singularities are exactly those with the *ADE* singularities;
4. Quotient singularities are rational;
5. Toric varieties have rational singularities.

The Du Bois complex

Let X be a projective complex algebraic variety.

$$X \text{ is smooth } \longrightarrow \text{Hodge structure on } H^n(X, \mathbb{C})$$

$$Gr_F^k H^n(X, \mathbb{C}) = H^{n-k}(X, \Omega_X^k)$$

$$X \text{ is singular } \longrightarrow \text{Mixed Hodge structure on } H^n(X, \mathbb{C})$$

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Du Bois cpx
constructed using
hypercvrs

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Theorem (Deligne 1973, Du Bois 1981)

For any complex algebraic variety X, the objects $\underline{\Omega}_X^k \in D^b_{coh}(X)$ is independent of the choice of hyperresolution for all k.

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Theorem (Deligne 1973, Du Bois 1981)

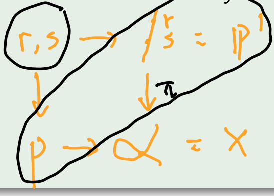
For any complex algebraic variety X, the objects $\underline{\Omega}_X^k \in D^b_{coh}(X)$ is independent of the choice of hyperresolution for all k.

For each k, we have a natural map $\Omega_X^k \to \underline{\Omega}_X^k$. It is a quasi-isomorphism when X is smooth.

Examples

Example (Node)

Let X be the nodal cubic $y^2z = x^2(x - z)$ in $\mathbb{P}^2$.

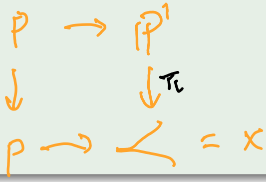


$$\underline{\Omega}^0_X = [\mathcal{O}_p \oplus \pi_*\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{r,s}] \overset{q.i.s}{\simeq} \mathcal{O}_X$$

$$\underline{\Omega}^1_X = [\Omega^1_p \oplus \pi_*\Omega^1_{\mathbb{P}^1} \to \Omega^1_{r,s}] \simeq \pi_*\Omega^1_{\mathbb{P}^1} \not\simeq \Omega^1_X$$

Example (Cusp)

Let X be the cuspidal cubic $y^2z = x^3$ in $\mathbb{P}^2$.



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Du Bois singularities

Definition (Steenbrink 1983)

We say X has *Du Bois singularities* if the natural map

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Rational singularities are Du Bois.

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Example

1. The nodal cubic has Du Bois singularities, but the cuspidal cubic does not;
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Higher Du Bois singularities

Let X be a local complete intersection.

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Definition (Jung-Kim-Saito-Yoon 2021)

We say X has *k-Du Bois singularities* if the natural morphism

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- 0-Du Bois singularities $\iff$ Du Bois singularities
- k-Du Bois $\implies$ $(k-1)$-Du Bois $\implies \cdots$

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Definition 1 (Friedman-Laza 2022)

A variety X has k-rational singularities if the natural map

$$\Omega_X^p \to \mathbf{R}\mathcal{H}om(\underline{\Omega}_X^{n-p}, \omega_X^\bullet)[-n]$$

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Definition 2 (Mustață-Popa 2022)

A variety X has k-rational singularities if for some (any) strong log resolution $f : \tilde{X} \to X$, the natural map

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Equivalent in the lci case. Not known to be equivalent in general.

Implications

Theorem (Saito 1993, Saito 2009, Jung-Kim-Saito-Yoon 2021, Mustață-Olano-Popa-Witaszek 2021, Mustață-Popa 2022, Chen-Dirks-Mustață 2023)

Let X be a local complete intersection of pure codimension r, and $\tilde{\alpha}_X$ its minimal exponent. Then

- *X has k-Du Bois singularities* $\iff \tilde{\alpha}_X \geq k + r$;
- *X has k-rational singularities* $\iff \tilde{\alpha}_X > k + r$.

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In particular, in the lci case

$$k\text{-rational} \implies k\text{-Du Bois} \implies (k-1)\text{-rational}$$

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In particular, in the lci case

$$k\text{-rational} \overset{\checkmark}{\Longrightarrow} k\text{-Du Bois} \overset{?!}{\Longrightarrow} (k-1)\text{-rational}$$

Does these hold true in general?

Lack of examples

Example

Let Z be the cone over the d-th Veronese embedding of $\mathbb{P}^r$. Then

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- Z is a toric variety;
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On the other hand,

- We don't have $\Omega_Z^1 \xrightarrow{\text{qis}} \underline{\Omega}_Z^1$ for $d \geq 2$.

⇓

$\Omega_Z^1 \not\cong \mathcal{H}^0 \underline{\Omega}_Z^1$

not reflexive　　reflexive

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No known example of non-lci varieties with 1-Du Bois singularities!

New definitions

Observation. The conditions

$$\Omega_X^p \xrightarrow{\text{qis}} \underline{\Omega}_X^p, \forall p \le k \iff \forall p \le k, \mathcal{H}^0\underline{\Omega}_X^p \cong \Omega_X^p \text{ and } \mathcal{H}^i\underline{\Omega}_X^p = 0 \text{ for all } i > 0$$

problematic

reflexive diff

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Definition

We say that X has *pre-k-Du Bois* singularities if

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$\iff R^i f_* \Omega^p_{\tilde{X}}(\log E) = 0$

Properties

Theorem (S.-Venkatesh-Vo)

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Corollary (S.-Venkatesh-Vo)

If X is a variety for which

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This extends results of Mustaţă-Popa (2022) and Friedman-Laza (2022) in the lci or isolated singularities cases.

Properties

Corollary (Friedman-Laza 2022, S.-Venkatesh-Vo)

If X is a normal projective variety with pre-k-rational singularities, then

$$\underline{h}^{p,q} = \underline{h}^{q,p} = \underline{h}^{n-p,n-q}$$

for any $0 \leq p \leq k$ and $0 \leq q \leq n$, where $\underline{h}^{p,q} := \dim \mathbb{H}^q(Y, \underline{\Omega}_X^p)$.

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Corollary (S.-Venkatesh-Vo)

Let $\pi : Y \to X$ be a finite dominant morphism of normal varieties and let Y have rational singularities. If Y has pre-k-Du Bois singularities, then X also has pre-k-Du Bois singularities.

In particular, quotient singularities are pre-k-Du Bois for all k.

Examples

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Example (Quotient singularities)

Quotient singularities are pre-k-Du Bois for all k (Du Bois 1981).

Cones over smooth projective varieties

Example (Cortiñas-Haesemeyer-Walker-Weibel 2010, S.-Venkatesh-Vo)

Let X be a smooth projective variety of dimension n, L an ample line bundle on X. The *affine cone over* X *with conormal bundle* L is the affine algebraic variety

$$C(X, L) = \operatorname{Spec} \bigoplus_{m \geq 0} H^0(X, L^m).$$

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$$C(X, L) = \operatorname{Spec} \bigoplus_{m \geq 0} H^0(X, L^m).$$

Then $C(X, L)$ has pre-k-Du Bois singularities if and only if

$$H^i(X, \Omega_X^p \otimes L^m) = 0 \text{ for all } i \geq 1,\ m \geq 1 \text{ and } p \leq k;$$

Cones over smooth projective varieties

Definition

A smooth projective variety $X \subset \mathbb{P}^N$ is said to *satisfy Bott vanishing* if

$$H^i(\Omega_X^k \otimes L) = 0$$

for all $i > 0$, $k \geq 0$ and any ample line bundle L on X.

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Example

The following varieties satisfy Bott vanishing: Projective spaces (Bott 1957), toric varieties (Danilov 1978, Steenbrink), abelian varieties, del Pezzo surfaces of degree ≥ 5 (Totaro 2020), K3 surfaces of Picard number 1 and degree 20 and ≥ 24 (Totaro 2020), stable GIT quotients of $\mathbb{P}^n$ by the action of PGL_2 (Torres 2020), a list of 37 Fano 3-folds (Totaro 2023), projective varieties with int-amplified endomorphism (Kawakami-Totaro 2023) etc.

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It follows that the cones over these varieties associated to any ample conormal bundle are pre-k-Du Bois for all k.

Cones over smooth projective varieties

Example (S.-Venkatesh-Vo)

- For $k \leq n$, the cone $C(X, L)$ has pre-k-rational singularities if and only if

$$H^i(X, \Omega_X^p \otimes L^m) = 0 \text{ for all } i \geq 1,\ m \geq 0 \text{ and } p \leq k,$$

except possibly when $m = 0$ and $i = p$, in which case

$$H^0(X, \mathcal{O}_X) \xrightarrow{\simeq} H^1(X, \Omega_X^1) \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} H^k(X, \Omega_X^k)$$

are isomorphisms via the map $c_1(L)$;

- If $C(X, L)$ has pre-n-ratonal singularities, then it has pre-$(n+1)$-rational singularities.

Higher rational and higher Du Bois singularities

Definition

Let X be a variety. X is said to have k-**rational singularities** if it is normal, and

1. X is pre-k-rational;
2. $\operatorname{codim}_X(X_{\text{sing}}) > 2k + 1$.

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Definition

Let X be a variety. X has k-**Du Bois singularities** if it is seminormal, and

1. X has pre-k-Du Bois singularities;
2. $\operatorname{codim}_X(X_{\text{sing}}) \geq 2k+1$;
3. $\mathcal{H}^0\underline{\Omega}^p_X$ is reflexive, for all $p \leq k$.

Properties of k-rational and k-Du Bois singularities

Proposition (S.-Venkatesh-Vo)

1. 0-*rational singularities* $\iff$ *rational singularities;*
   0-*Du Bois singularities* $\iff$ *Du Bois singularities.*

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1. 0-*rational singularities* $\iff$ *rational singularities;*
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2. *When* X *is lci, these agree with the existing definitions.*

Properties of k-rational and k-Du Bois singularities

Proposition (S.-Venkatesh-Vo)

1. *0-rational singularities* $\iff$ *rational singularities;*
   0-Du Bois singularities $\iff$ *Du Bois singularities.*
2. *When X is lci, these agree with the existing definitions.*

Theorem (S.-Venkatesh-Vo)

If a variety X has k-rational singularities, then it has k-Du Bois singularities.

Examples

Example

Let X be an affine toric variety and let $c := \mathrm{codim}_X(X_{\mathrm{sing}})$. Then:

1. X has k-Du Bois singularities for all $0 \leq k \leq \frac{c-1}{2}$;
2. If X is simplicial, then it has k-rational singularities for all $0 \leq k < \frac{c-1}{2}$;
3. If X is non-simplicial, then it does not have pre-1-rational singularities, hence it does not have 1-rational singularities.

Example

The cone $C(X, L)$ is k-Du Bois if and only if it is pre-k-Du Bois, $k \leq n/2$ and

$$H^i(X, \mathcal{O}_X) = 0 \text{ for all } 0 < i \leq k.$$

It is k-rational if and only if it is pre-k-rational and $k < n/2$.

k-DB $\not\Rightarrow$ (k-1)-rational

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It is k-rational if and only if it is pre-k-rational and $k < n/2$.

Example

1. The cone $Z = C(\mathbb{P}^r, \mathcal{O}(d))$ has k-Du Bois singularities if and only if $k \leq \frac{r}{2}$. It is k-rational if and only if $k < \frac{r}{2}$.
2. The cone $Z = C(X, \mathcal{O}(d))$ over a quartic surface $X \subset \mathbb{P}^3$ is not 2-Du Bois, and is 1-Du Bois if and only if $d \geq 5$. However, Z does not have rational singularities.

Proof that pre-k-rational $\implies$ pre-k-Du Bois

Case X is projective. Note that we have natural maps

$$\mathcal{H}^0\underline{\Omega}_X^k \xrightarrow{f} \underline{\Omega}_X^k \xrightarrow{g} \mathbf{R}\mathcal{H}om(\underline{\Omega}_X^{n-k}, \omega_X^\bullet)[-n].$$

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1 Since X is normal and pre-k-rational, the composition $g \circ f$ is a quasi-isomorphism. It follows that f induces injective map

$$H^i(X, \mathcal{H}^0\underline{\Omega}_X^k) \to \mathbb{H}^i(X, \underline{\Omega}_X^k) \qquad (*)$$

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2 By Hodge theory, we have surjections

$$H^i(X, \mathcal{H}^0\underline{\Omega}_X^{\leq k}) \to \mathbb{H}^i(X, \underline{\Omega}_X^{\leq k})$$

for all i and k. If X is pre-$(k-1)$-Du Bois (which we can assume by induction), then $(*)$ is surjective for all i;

3 Combining (1) and (2), we see that $(*)$ is an isomorphism. This implies

$$\mathcal{H}^0\underline{\Omega}_X^k \xrightarrow{\text{qis}} \underline{\Omega}_X^k$$

(i.e. X is pre-k-Du Bois), provided the "bad locus"

$$\Sigma_X := \operatorname{supp} \operatorname{Cone}(\mathcal{H}^0\underline{\Omega}_X^k \to \underline{\Omega}_X^k)$$

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The general case. When X is not necessarily projective, we don't have the surjection given by Hodge theory. The trick, due to Kovács, is to compactify X and use excision of local cohomology to get rid of the contribution from the boundary of the compactification.

Thank you for listening!